Root Numbers of Abelian Varieties

Matthew Bisatt

King's College London

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Corollary

If $W(A/K) = -1$, then $|A(K)|$ is infinite.

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- $W(A/K_v) = (-1)^{\dim A}$ *if* $v|\infty$*.*

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So $E(\mathbb{Q})$ is infinite, assuming the parity conjecture.

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$$W(A/\mathbb{Q}_p) = (-1)^{\langle \rho_T, \mathbb{1} \rangle} \left(\frac{-1}{p} \right)^{m_T},$$

where m_T *is the multiplicity of* -1 *as an eigenvalue for* $\rho_T(\iota)$*, with* ι *any generator of the tame inertia group.*

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$$W(E/\mathbb{Q}_p) = \begin{cases} 1 & \textit{if } e = 1 \textit{ (good reduction)}; \\ \left(\dfrac{-1}{p}\right) & \textit{if } e = 2, 6; \\ \left(\dfrac{-3}{p}\right) & \textit{if } e = 3; \\ \left(\dfrac{-2}{p}\right) & \textit{if } e = 4. \end{cases}$$

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$$W_{p,e} = \begin{cases} \left(\frac{p}{l}\right) & \text{if } e = l^k; \\ \left(\frac{-1}{p}\right) & \text{if } e = 2l^k \text{ and } l \equiv 3 \mod 4, e = 2; \\ \left(\frac{-2}{p}\right) & \text{if } e = 4; \\ \left(\frac{2}{p}\right) & \text{if } e = 2^k \text{ for } k \geq 3; \\ 1 & \text{else,} \end{cases}$$

where $k > 0$ and l is any odd prime.

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Theorem (B.)

Let $p > 2\dim A + 1$. Using the notation above:

$$W(A/\mathcal{K}) = (-1)^{\langle \rho_T, \mathbb{1} \rangle} W_{p,2}^{m_T f(\mathcal{K}/\mathbb{Q}_p)} \left(\prod_{e \in \mathbb{N}} W_{p,e}^{m_e} \right)^{f(\mathcal{K}/\mathbb{Q}_p)}.$$

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Let $C/\mathbb{Q} : y^2 = x^6 - 10x^4 + 2x^3 + 21x^2 - 18x + 5,\ N = 103^2$.

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Lemma (Dokchitser–Dokchitser–Maistret–Morgan)

Let $C/\mathbb{Q}_p$ be a hyperelliptic curve with $p > 2\,\mathrm{genus}(C) + 1$. Then the inertia representation $\rho^*_{\mathrm{Jac}(C)}$ attached to $H^1_{ét}(C/\overline{\mathbb{Q}}_p, \mathbb{Q}_l) \otimes_{\mathbb{Z}_l} \mathbb{C}$ is computable from a Weierstrass model.

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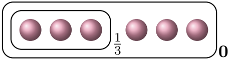

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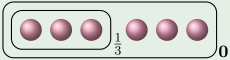

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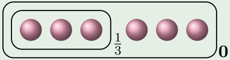

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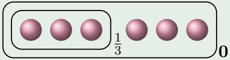

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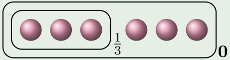

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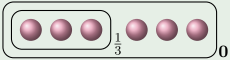

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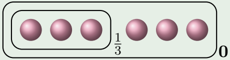

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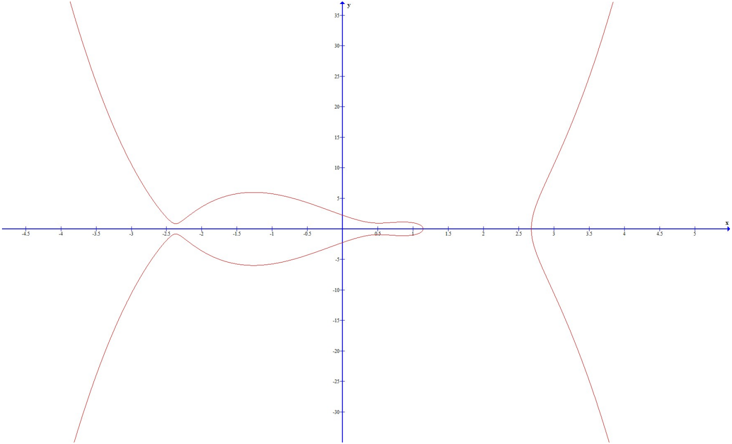
Remark

The Mordell-Weil rank of $\mathrm{Jac}(C/\mathbb{Q})$ (conjecturally) increases in every quadratic extension $K/\mathbb{Q}$.

Thanks for listening!!



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